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Kleinian groups and symmetry of domain structures. Domain branching in superconductors and magnetics

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Abstract

The domain structure of a scalar field or a vector potential and a solenoidal field on a plane is shown to have a symmetry of the Kleinian group. This allows us to build a classification of domain structures for superconductors and magnetics by means of the Kleinian groups and explain their main properties. A number of examples of the domain branching on a plane boundary of superconductors and magnetics are described. Generalizations of the theory allow us to take in account the more general types of field, the symmetry of the crystal lattice, the three-dimensionality of space and more general functionals of the free energy.

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Introduction

Physical fields in solids are known to form different domain structures. We show in this paper that similar to the description of the crystal symmetry by space groups the symmetry of domain structures is described by the Kleinian groups. This allows us to build a classification of domain structures for superconductors and magnetics and explain their main properties.

In this paper we study mainly the domain branching phenomenon, i.e. the hierarchical, self-similar and fractal splitting up of a domain of the field on a boundary of a solid. The domain branching in superconductors was first studied by Landau [11, 12]. A consideration of the domain branching in magnetics was started by Landau and Lifshitz [10], Lifshitz [13] and especially by Privorotsky [14, 15].

The aim of this paper is to describe the domain branching phenomenon by a language of symmetry groups and complex analysis.

The main idea is as follows. As is well known, any analytical function is a mathematical image of the plane scalar field or the complex potential of a potential and solenoidal vector field on a plane. We know also that the only possible conformal transformations of the complete complex plane into itself are the linear-fractional transformations which form the Möbius group. Therefore any symmetry of the field on a plane is described by a discontinuous subgroup or, which is the same thing, by a Kleinian subgroup of the Möbius group M_2, and the domains of the field must correspond to fundamental domains of this subgroup. We apply this idea to discussion of the domain branching in superconductors and magnetics.

The organization of the paper is the following. In the first section we build a theory of domain structures in superconductors and on a simple example study the domain branching on a plane boundary of superconductors. In the second section we consider domain structures in magnetics in the simplest situation. The domain branching in magnetics is shown to lead to a natural width of the domain boundary and in such a way to determine the limitations for the usage of magnetic domains for storage and reproducing of information. Then we present the general formulation of our approach to the domain branching phenomenon and consider different generalizations of the theory with the aim of taking into account the more general types of field, the symmetry of the crystal lattice, the three-dimensionality of space and more general functionals of the free energy.

1. Scalar order parameter. Domains in superconductors

Let us consider a plane superconductor and let us describe its points by real coordinates x, y or complex numbers $z = x + \imath y$ where $\imath$ is the imaginary unit. The superconductor is well known to be characterized by a complex function (the London order parameter) $\Psi(x, y) = \Psi(z)$, which has the following physical meaning: $|\Psi(z)|$ is the spectral gap for the one-electron excitations in the superconductor and grad arg $\Psi(z)$ is the current.

The order parameter is an extremal of the Ginzburg–Landau functional of non-equilibrium free energy

$$F = \int \{|\mathrm{grad}\Psi|^2 + P_4(|\Psi|)\}\,\mathrm{d}x\,\mathrm{d}y \qquad P_4(|\Psi|) = (a/2)|\Psi|^2 + (b/4)|\Psi|^4 \tag{1.1}$$

and therefore satisfies the equation

$$\Delta\Psi = a\Psi + |\Psi|^2\Psi. \tag{1.2}$$

If we can neglect the polynomial of fourth order $P_4(|\Psi|)$, which is valid near the phase transition temperature, then the Ginzburg–Landau functional transforms to the Dirichlet functional

$$D = \int |\mathrm{grad}\Psi|^2\,\mathrm{d}x\,\mathrm{d}y \tag{1.3}$$

and hence the order parameter appears to be a solution of the Laplace equation

$$\Delta\Psi = 4\frac{\partial^2\Psi}{\partial z\partial\bar{z}} = 0. \tag{1.4}$$

A general solution of the Laplace equation is a sum of analytical and anti-analytical functions. Furthermore, we shall study a particular solution which is an analytical function, $\Psi(z)$.

Let us recall that the Laplace equation is invariant with respect to conformal transformations $\zeta = f(z)$, $f'(z) \neq 0$:

$$\frac{\partial^2\Psi}{\partial z\partial\bar{z}} = |f'(z)|^2\frac{\partial^2\Psi}{\partial\zeta\partial\bar{\zeta}} = 0 \qquad \zeta = f(z). \tag{1.5}$$

Let us recall also that the only possible conformal transformations of the extended complex plane $\overline{\mathbb{C}} = \mathbb{CP}$ into itself are the linear-fractional transformations

$$\zeta = \gamma z = \frac{az + b}{cz + d} \qquad ad - bc = 1 \qquad (a, b, c, d \in \mathbb{C}) \tag{1.6}$$

which form the Möbius group M_2. Taking into account a natural isomorphism

$$M_2 \cong SL(2, \mathbb{C})/\{\pm 1_2\} \tag{1.7}$$

where 1_2 is the 2×2 unit matrix, we shall write the elements γ of the Möbius group in the form

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{1.8}$$

Thus it follows from the consideration presented above that the domain structures of the superconductor correspond to discontinuous subgroups or, which is the same thing, to the Kleinian subgroups of the Möbius group M_2, and the domains of the superconductor correspond appropriately to fundamental domains of these subgroups.

The simplest example of the Kleinian groups is principal congruence subgroups of the level N

$$\Gamma(N) = \{\gamma \in M_2 | \gamma \equiv 1_2 \bmod (N)\} \tag{1.9}$$

where N is a positive integer.

The order parameter in a superconductor is scalar. This means that in a superconductor with a domain structure the order parameter $\Psi(z)$ must be the automorphic function of the Kleinian group Γ characterizing the symmetry of the appropriate domain structure. In other words the order parameter $\Psi(z)$ must satisfy the functional equation

$$\Psi(\gamma z) = \Psi(z) \qquad z \in \overline{\mathbb{C}} \qquad \gamma \in \Gamma.$$

If we take into account a boundary of the superconductor then the elements of the Kleinian group must leave this boundary invariant. We recall in this connection that if the Kleinian group Γ leaves invariant some circle (or line) we call it the Fuchsian group. Therefore the Kleinian group describing the domain structure of a bounded superconductor must be the Fuchsian group.

Example. The domain structure in a superconductor, described by the principal congruence subgroup $\Gamma(2)$. We consider a superconductor in the upper half plane and designate points of the upper half plane $\mathbb{C}^+$ by $\tau = \eta + \imath\zeta$.

The order parameter $\mu(\tau)$ of the superconductor will be the automorphic function of the principal congruence subgroup of the level 2 $\Gamma(2)$.

The generators of this subgroup are elements

$$\gamma_1 : \gamma_1\tau = \tau - 2 \qquad \gamma_2 : \gamma_2\tau = \frac{\tau}{-2\tau + 1}.$$

Every element of $\Gamma(2)$ is a product of a finite number of powers of generators γ_1, γ_2:

$$\gamma = \gamma_1^{m_1}\gamma_2^{n_1} \ldots \gamma_1^{m_k}\gamma_2^{n_k}.$$

We can define the fundamental domain $F(2)$ of the subgroup $\Gamma(2)$ in the following way:

$$F(2) = \{\tau | \operatorname{Im}\tau > 0, -1 \leqslant \operatorname{Re}\tau < 1, |\tau - 1/2| > 1/2, |\tau + 1/2| \geqslant 1/2\}. \tag{1.10}$$

Using the Eisenstein series it is easy to show that the subgroup $\Gamma(2)$ has the following automorphic function:

$$\begin{aligned} \mu(\tau) = 1 - \lambda(\tau) &= \frac{e_1(\tau) - e_2(\tau)}{e_1(\tau) - e_3(\tau)} = \frac{\wp(\tau, \imath\pi) - \wp(\tau, \imath\pi\tau)}{\wp(\tau, \imath\pi) - \wp(\tau, \imath\pi(1+\tau))} \\ &= \frac{\theta_3^4(0|\tau) - \theta_2^4(0|\tau)}{\theta_3^4(0|\tau)} = 1 - 16\left(\frac{\sum_{m=0}^{\infty} q^{(m+1/2)^2}}{1 + 2\sum_{m=1}^{\infty} q^{m^2}}\right)^4 \\ &= 1 - 16q\prod_{m=1}^{\infty}\left(\frac{1 - q^{2m}}{1 + q^{2m-1}}\right)^8 \qquad q = \mathrm{e}^{\imath\pi\tau} \qquad |q| < 1. \end{aligned} \tag{1.11}$$

Here we use standard notations of a theory of elliptic modular functions. For example,

$$\wp(\tau, x) = \frac{1}{x^2} + \sum_{m,n}{}' \left[\frac{1}{(x - m - n\tau)^2} - \frac{1}{(m + n\tau)^2} \right]$$

is the Weierstrass function.

Let us enumerate the main properties of the function $\mu(\tau)$:

(1) The function $\mu(\tau)$ is analytical in the upper half plane and does not have there the values $0, 1, \infty$. On the real line $\mu(\infty) = 1$, $\mu(\pm 1) = \infty$, $\mu(0) = 0$ and the 0 is of the first order.

(2) $\mu(\tau)$ is an automorphic function of the principal congruence subgroup of the level 2

$$\Gamma(2) = \{\gamma \in M_2 | \gamma \equiv 1_2 \bmod (2)\}. \tag{1.12}$$

This means that

$$\mu(\gamma\tau) = \mu(\tau) \qquad \gamma \in \Gamma(2).$$

Every simple automorphic function of the subgroup $\Gamma(2)$ is a rational function of $\mu(\tau)$.

(3) The function $\mu(\tau)$ maps the fundamental domain $F(2)$ on a complex plane with a cut from zero to infinity in such a way that the right-hand part of $F(2)$ goes to the upper half plane of μ. The function $\mu(\tau)$ accepts every value in the fundamental domain $F(2)$ of the principal congruence subgroup $\Gamma(2)$ once and only once. $\mu(-\overline{\tau}) = \overline{\mu}(\tau)$; i.e., the function $\mu(\tau)$ is symmetric with respect to the imaginary axis. The function $\mu(\tau)$ has real values on the imaginary axis and on the boundary of the fundamental domain $F(2)$.

The fundamental domain $F(2)$ of the principal congruence subgroup $\Gamma(2)$ consists of six fundamental domains $F(1)$ of $\Gamma(1)$. Under the action of the modular group $\Gamma(1)$ the function $\mu(\tau)$ is transformed to

$$\mu(\tau) \qquad 1 - \mu(\tau) \qquad \frac{1}{\mu(\tau)} \qquad \frac{1}{1 - \mu(\tau)} \qquad \frac{\mu(\tau)}{1 - \mu(\tau)} \qquad \frac{1 - \mu(\tau)}{\mu(\tau)}.$$

These six transformations form the group of anharmonic quotients.

(4) The function inverse to the function $\mu(\tau)$ has the form

$$\tau(\mu) = i \frac{{}_2F_1(1/2, 1/2; 1; \mu)}{{}_2F_1(1/2, 1/2; 1; 1 - \mu)} \tag{1.13}$$

where ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function.

Thus in this example we have a superconductor in the upper half plane with the order parameter having the value unity at infinity and a dense set of values zero and infinity at a real line. Of course we should not take into account seriously the infinite values of the order parameter since at appropriate points we cannot neglect in the free energy functional of the superconductor (1.1) the polynomial of the fourth order. The zero values of the order parameter are centres of magnetic vortices which are perpendicular to the plane. Hence we have the superconductor bounded with a dense set of magnetic vortices, which create currents, preventing the penetration of magnetic field inside the superconductor. The real line is a natural boundary of existence for the analytical function $\mu(\tau)$. This is obvious since the real line is the boundary of the superconductor itself and therefore the order parameter of the superconductor may exist only inside the superconductor.

As a domain in the superconductor we can choose the fundamental domain $F(2)$. The images of this domain, $\gamma F(2)$, $\gamma \in \Gamma(2)$, are diminishing with increasing order of the mapping and thus describe the domain branching at the boundary of the superconductor.

Above we have studied the domain branching of the scalar order parameter at the boundary of a half-plane. Of course we can carry over these results to a disc (by a linear-fractional map), a stripe (by a trigonometric function), a rectangle (by an elliptic function map) etc.

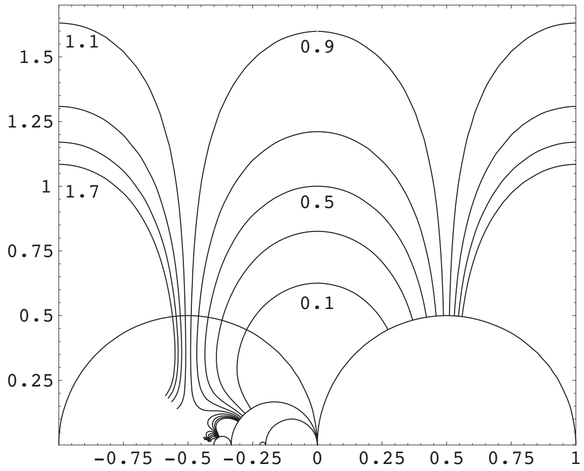

Figure 1. The upper half complex plane with the fundamental domain of the principal congruence subgroup $\Gamma(2)$ and the lines $|\mu| = \text{const}$ of the automorphic function $\mu(z)$ defined by the formula (1.11).

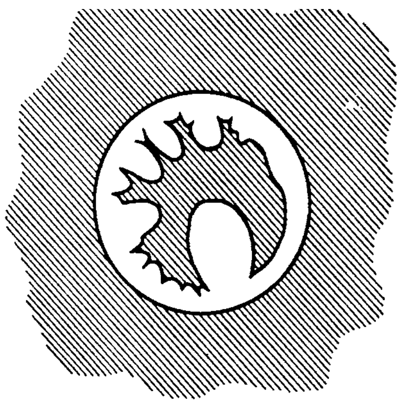

Figure 2. Experimental data for the magnetic field penetration into a superconductor of the second type (Nb–Ta alloy) presented in the book [16].

In figure 1 in the upper half plane we can see the fundamental domain of the superconductor with its images and the lines $|\mu| = \text{const}$ calculated by means of the expression (1.13). We can consider the line $|\mu| = \epsilon$, where ϵ is a given positive number, as a boundary between the regions of the superconducting and normal states. Thus we see that the magnetic field penetrates into the superconductor in the form of 'fingers'.

This theoretical result is illustrated by figure 2, borrowed from the book [16], which represents results of experimental studies of the magnetic field penetration into a superconductor of the second type (Nb–Ta alloy). We can see a lot of 'fingers' of different size. The last property is a direct consequence of the conformal invariance of our model when a length unit is arbitrary.

Above we have studied the domain structure described by the symmetry group $\Gamma(2)$. It is possible to show that the modular group $\Gamma(1)$ and the appropriate modular function $J(\tau)$ describe the situation when the magnetic field has already penetrated into the superconductor and centres of magnetic vortices form an ordered structure inside the superconductor. We think that on changing the external magnetic field the superconducting order parameter will change its symmetry and will go over from one structure to another by means of a sequence of phase transitions of second order.

The results presented above are valid for any plane scalar field and in such a way are related to the so-called Laplace fractals and the problems of growth.

2. Vector order parameter. Domains in magnetics

As an example of a vector field on a plane let us consider the magnetic induction, which we shall designate by a vector $\vec{B} = (B_x, B_y)$ or a complex number $B(z) = B_x + \imath B_y$. It is well known that the magnetic induction satisfies the following equations:

$$\mathrm{div}\vec{B} = 0 \qquad \mathrm{rot}\,\vec{B} = 4\pi\,\mathrm{rot}\,\vec{M}$$

where $\vec{M}$ is a magnetization. If the magnetization is a potential field, $\mathrm{rot}\vec{M} = 0$, then the magnetic induction $B(z)$ satisfies the equation

$$\mathrm{div}\vec{B} = 0 \qquad \mathrm{rot}\vec{B} = 0 \tag{2.1}$$

and therefore has a complex potential $\Psi(z) = u(z) + \imath v(z)$,

$$B(z) = \frac{\mathrm{d}}{\mathrm{d}z}\overline{\Psi(z)}. \tag{2.2}$$

This means that the magnetic induction $B(z)$ can be presented in the following way:

$$\vec{B} = \mathrm{grad}\, u = \mathrm{rot}\,\vec{v} \qquad \text{where} \quad \vec{v} = (0, 0, v). \tag{2.3}$$

The plane vector field $B(z)$ can be presented by the differential form $F(z)\,\mathrm{d}z$. The integral of this form along a closed boundary of any domain D,

$$\begin{aligned}\int_{\partial D} \overline{B}(z)\,\mathrm{d}z &= \int_D \mathrm{rot}\,\vec{B}(x, y)\,\mathrm{d}x\,\mathrm{d}y + \imath \int_D \mathrm{div}\vec{B}(x, y)\,\mathrm{d}x\,\mathrm{d}y \\ &= 2\pi\imath \sum_{a_k \in D} \mathrm{res}_{a_k}\overline{B}(z) = C + \imath Q\end{aligned} \tag{2.4}$$

has clear physical meaning: Q is the intensity and C is the circulation of the vector field $B(z)$ at the singular points a_k, $k = 1, \ldots$.

The form $B(z)\,\mathrm{d}z$ is invariant with respect to conformal transformations of coordinates, $B(z)\,\mathrm{d}z = B(\sigma(z))\,(\mathrm{d}\sigma(z)/\mathrm{d}z)\,\mathrm{d}z$, and if the vector field $B(z)$ has a domain structure then, according to the discussion above, its symmetry is characterized by some Kleinian subgroup Γ of the Möbius group of linear-fractional transformations.

Let us recall that $f(z)$ is called the Γ-automorphic form of the weight k if

$$f(\sigma(z))(cz + d)^{-k} = f(z) \qquad \sigma(z) = (az + b)/(cz + d). \tag{2.5}$$

Let $A_k(\Gamma)$ be a set of automorphic forms, $G_k(\Gamma)$ a set of holomorphous forms and $S_k(\Gamma)$ a set of parabolic forms.

For the vector field we have

$$B(z)\,\mathrm{d}z = B(\sigma(z))\frac{\mathrm{d}\sigma(z)}{\mathrm{d}z}\,\mathrm{d}z$$

and therefore

$$B(z) = B(\sigma(z))(cz + d)^{-2}. \tag{2.6}$$

Thus the vector field $B(z)$ is the Γ-automorphic form of the weight 2, $B(z) \in A_2(\Gamma)$. We shall assume further that the vector field $B(z)$ is a holomorphic form, $B(z) \in G_2(\Gamma)$.

Therefore it is important to know for every Kleinian group Γ the dimension of a linear space of Γ-automorphic form of weight 2. It is well known (see, e.g., [6, 17, 18]) that

$$\dim G_2(\Gamma) = \begin{cases} g + m - 1 & m > 0 \\ g & m = 0. \end{cases}$$
$$\dim S_2(\Gamma) = g$$

where g is the genus of the Riemann surface $\Gamma \setminus \mathbb{C}_+$, $\mathbb{C}_+ = \{z \in \mathbb{C} | \mathrm{Im}\, z > 0\}$ and m is a number of non-equivalent parabolic points of the group Γ. For the principal congruence subgroups

$$\Gamma(N) = \{\sigma \in \Gamma(1) | \sigma = 1_2, \ \mathrm{mod}\ (N)\}$$

it appears as follows: for $\Gamma(1)$ we have $g = 0$ and $m = 1$; for $\Gamma(2)$ we have $g = 0$ and $m = 3$, since in this case there are three non-equivalent parabolic points 0, 1, ∞, and so on. Therefore for $\Gamma(N)$ we have the following result:

$$\dim G_2(\Gamma(1)) = 0 \qquad \dim G_2(\Gamma(2)) = 2, \ \text{etc.}$$

We consider further the domain structure of vector field $B(z)$ characterized by the Kleinian group $\Gamma(2)$ since in this case $\dim G_2(\Gamma(N))$ has the least possible non-zero value.

Example. The domain structure in a magnetic, described by the principal congruence subgroup $\Gamma(2)$.

We begin with the following result:

$$\dim G_2(\Gamma(2)) = 2 \qquad \dim S_2(\Gamma(2)) = 0.$$

Using the Eisenstein series it is easy to show that two linearly independent holomorphic forms in the two-dimensional space $G_2(\Gamma(2))$ are

$$f_1(\tau) = -6e_1(\tau) = -6\wp(\tau, \pi\imath) = 1 + 24 \sum_{n=1}^{\infty} \sigma_1^{\mathrm{odd}}(n) q^n \tag{2.7}$$

$$f_2(\tau) = 2e_2(\tau) = 2\wp(\tau, \pi\imath\tau) = 1 + 24 \sum_{n=1}^{\infty} \sigma_1^{\mathrm{odd}}(n) q^{n/2} \tag{2.8}$$

where

$$\sigma_1^{\mathrm{odd}}(n) = \sum_{d|n, d\equiv 1 \ \mathrm{mod}\ (2)} d$$

and $q = \mathrm{e}^{2\pi\imath\tau}$.

Thus any holomorphic form in the two-dimensional space $G_2(\Gamma(2))$ is a linear superposition of the forms $f_1(\tau)$, $f_2(\tau)$.

It follows straight from the definition that

$$f_1((1 + \imath)/2) = 0 \qquad f_2(-1 + \imath) = 0 \tag{2.9}$$

i.e. the points $(1 + \imath)/2$, $(-1 + \imath)$ are critical points of the vector fields $f_1(\tau)$, $f_2(\tau)$. We can consider any critical point as a point of 'collision' of a domain and an anti-domain of the vector field. By means of the linear-fractional transformations $\gamma \in \Gamma(2)$ these points are multiplied and form a dense set near the boundary.

Since forms $f_1(\tau)$, $f_2(\tau)$ are related by a simple affine transformation we consider further only the first one.

In figure 3 in the upper half plane we show the fundamental domain of the magnetic with its images and the magnetic lines $f_1(\tau)$ calculated by means of the expression (2.7).

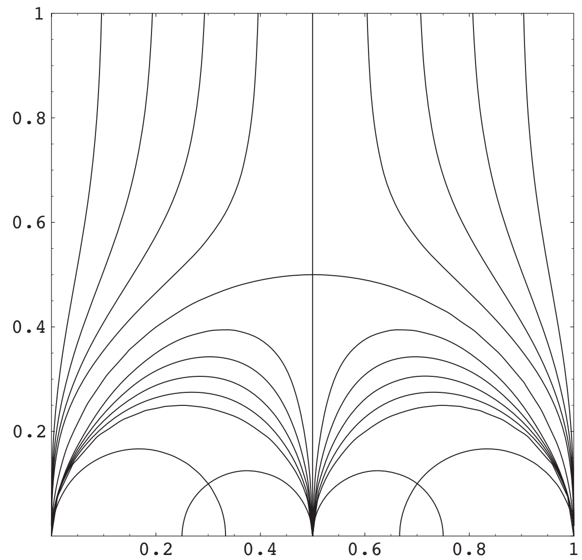

Figure 3. The upper half complex plane with the fundamental domain of the principal congruence subgroup $\Gamma(2)$ and the vector field corresponding to the $\Gamma(2)$-automorphic holomorphic form $f_1(\tau)$ of the weight 2 defined by the formula (2.7).

It is instructive to compare this exact theoretical domain structure with the approximate domain structures proposed by Landau and Lifshitz (see figure 1 in [10]) and Privorotskii [14]. The Landau–Lifshitz domain structure, which is observed, for example, in iron, has no branching and its critical points are situated on the boundary of a solid, but a global picture of magnetic lines is similar to that presented in our figure 3. In figure 4 we have reproduced the Privorotskii domain structure, which has a lower energy than the Landau–Lifshitz structure. The critical points in the Privorotskii domain structure are situated inside the solid, as in our case, but the picture of magnetic lines near the critical points differs essentially from our figure 3. We believe that our picture of magnetic lines near the critical points is more reliable because it is a straight consequence of the analyticity of the complex potential.

It is important to remark that the domain structures in magnetics have been studied in many papers, reviews and books (see e.g. [10, 13, 16]) but only by means of the straight variational methods applied to the free energy functionals.

In conclusion we wish to point out that to any discrete subgroup G of the Möbius transformations acting on the complete complex plane (or the Riemann sphere) we can refer the Poincaré exponent (or critical exponent) [1, 5]

$$\delta(G) = \inf\left\{s : \sum_{g\in G} \exp(-s\rho(0, g(0))) < \infty\right\}$$

where ρ is the hyperbolic metric. The limit set of the subgroup G, designated by $\Lambda(G)$, has either 0, 1, 2 or infinitely many points. The point $x \in \Lambda(G)$ is called a conical limit point if there exists a sequence of orbit points which converges to x inside a non-tangential cone with vertex at x. We denote a set of conical points by $\Lambda_c(G)$. It appears that if G is a non-elementary discrete Möbius subgroup then

$$\delta(G) = \dim(\Lambda_c(G))$$

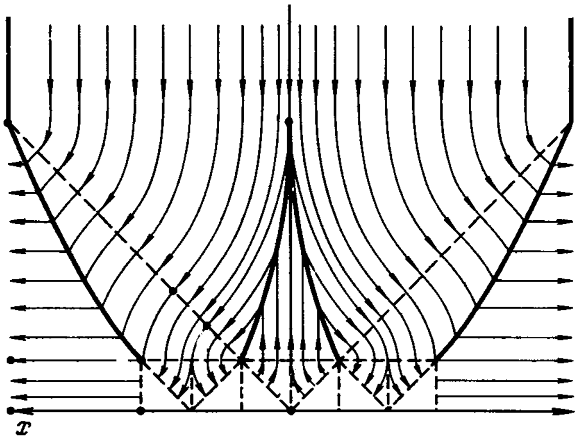

Figure 4. A domain of the magnetic structure with the appropriate magnetic lines presented in the paper [14].

where dim means the Hausdorff–Besicovich dimension (or fractal dimension) [3]. This dimension characterizes the natural width of boundaries of the domain structure and because of this it bears a relation to the accuracy of information recording by magnetic devices.

3. Formulation of the result and the generalizations

In previous sections we have studied some examples of domain structures in superconductors and magnetics. For illustration we have considered only $\Gamma(N)$ groups but of course there are no problems in studying more general Kleinian groups (see many examples in [8]).

Generalizing our observations we formulate the following statement:
symmetry structures of the plane scalar field or the vector potential and solenoidal field on a plane must correspond to discontinuous subgroups or, which is the same thing, to the Kleinian subgroups of the Möbius group, and the domains of the appropriate field are to be the fundamental domains of these subgroups.

Besides the scalar and vector fields we can also discuss an arbitrary tensor field. Below we consider some other generalizations of the theory presented above which give us the possibility to take in account the symmetry of the crystal lattice and non-homogeneity of the medium, the three-dimensionality of space and more general functionals of the free energy.

3.1. Quasi-conformal functions and deformations

It is possible to consider besides the analytical functions, which describe conformal maps, the quasi-analytical functions, which describe quasi-conformal maps, and therefore the quasi-conformal deformations of the Kleinian (and Fuchsian) groups [7]. In such a way we can take into account, for example, the crystal symmetry of a solid (see e.g. the second part of the book [9]).

The analytical and anti-analytical functions $\Psi(z)$ and $\Psi(\overline{z})$ satisfy appropriately the equations $\partial_{\overline{z}}\Psi(z) = 0$ and $\partial_z\Psi(\overline{z}) = 0$. A quasi-conformal map Ψ of a domain D satisfies the Beltrami equation

$$\partial_{\overline{z}}\Psi - \mu(z)\partial_z\Psi = 0$$

where $\mu(z)$ is a measurable function in D and $\|\mu\|_\infty < 1$.

For mappings of the Riemann surphases the function $\mu(z)$ must satisfy the equality $\mu(z)\,\mathrm{d}\bar{z}/\mathrm{d}z = \mu(z')\,\mathrm{d}\bar{z}'/\mathrm{d}z'$, where z, z' are local parameters of the surphases. If we consider quasi-conformal automorphisms f of the complete complex plane $\overline{\mathbb{C}}$ which are consistent with the Kleinian group G, i.e. such that the group $G_f = fGf^{-1}$ is a Kleinian one, then the form $\mu(z)\,\mathrm{d}\bar{z}/\mathrm{d}z$ must be G-invariant. Thus we obtain the isomorphisms $\chi : G \to G_f$ which are the quasi-conformal deformations of the group G.

3.2. Kleinian groups and domains in $\mathbb{R}^n$

It is possible to study the Kleinian groups in $\mathbb{R}^n$ [2]. Poincaré proposed to continue the action of an arbitrary Kleinian group to the upper half-space

$$\mathbb{R}^3_+ = \{(x, y, t) \in \mathbb{R}^3 | z = x + \imath y, t > 0\}$$

by means of inversions with respect to half-spheres with their centres situated on the complex plane $\mathbb{C}$.

We can fulfil this continuation by means of the quaternions. If we identify the complex number $z = x + \imath y \in \mathbb{C}$ with the quaternion $x + \imath y + j0 + k0 \in \mathbb{H}$ and the point $(x, y, t) \in \mathbb{R}^3_+$ with the quaternion $z + jt = x + \imath y + jt + k0 \in \mathbb{H}$ then we can define the action of the unimodular matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{C})$$

in the upper half-space $\mathbb{R}^3_+$ in the following way:

$$(z + jt) \to (z' + jt') = [a(z + jt) + b][c(z + jt) + d]^{-1}.$$

After continuation the elements of the Kleinian group act in $\mathbb{R}^3_+$ discontinuously and become non-Euclidean motions if we introduce the Poincaré metric

$$\mathrm{d}s^2 = (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}t^2)/t^2.$$

We can also consider the Möbius group M_n of all conformal automorphisms of the extended Euclidean space $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \infty$, $n \geqslant 3$. The definition of the Kleinian groups in this case is similar to that for the two-dimensional case. The Kleinian group is called the Fuchsian one if there exists a n-dimensional ball in $\overline{\mathbb{R}}^n$ which is invariant with respect to the group. The Kleinian group is called the quasi-Fuchsian one if there exists a Jordan surface in $\overline{\mathbb{R}}^n$ with interior and exterior which are homeomorphic to the ball and are invariant with respect to the group.

There also exist many-dimensional quasi-conformal mappings of manifolds and Kleinian groups although their set is quite restricted.

3.3. Conformal field theory and statistical mechanics

So far we have studied the quadratic free energy functionals which are valid near the temperature of the phase transition. It is possible to consider more general non-quadratic functionals in the frame of conformal field theory, which is used in physics to describe phase transitions. Using the modular covariance of characters of the Virasoro and Kac–Moody algebras (see e.g. [4,5]), we can construct non-quadratic free energy functionals which give us the possibility to describe the domain structures far from the temperature of the phase transition.

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